

# Ultrafilters as Propositional Theories

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Open up a (sufficiently advanced) mathematics textbook to the right page, and you can find a definition like the following:

## Definition 1.

Let  $I$  be a non-empty set. A **filter over  $I$**  is a non-empty set  $\mathcal{F} \subseteq \mathcal{P}(I)$  that does not contain  $\emptyset$  and which meets the following conditions:

- (i) If  $S, T \in \mathcal{F}$ , then  $S \cap T \in \mathcal{F}$ , and
- (ii) If  $S \in \mathcal{F}$  and  $S \subseteq T \subseteq I$ , then  $T \in \mathcal{F}$ .

Furthermore, if  $\mathcal{F}$  is also complement-complete, then it is an **ultrafilter**.

If you're at all like me, you spend a good couple hours trying to get a sense of how this definition works and why it allows us to prove interesting things before eventually giving up and seeing what's new on Netflix.

No surprise; mathematics texts are a genre unto themselves, and one most philosophers haven't been trained to read. But it is also unfortunate. Filters and ultrafilters are powerful technical tools that philosophers can put to good use. Philosophically accessible introductions to other technical tools aren't hard to find, but somehow filters and ultrafilters have been overlooked.<sup>1</sup> As a result, they remain a common gap in the technically-minded philosopher's toolkit.

My goal here is to fill that gap, presenting ultrafilters and theorems about them using philosophically familiar ideas. While some philosophers might encounter ultrafilters while studying topology, I'll present them as used in another of their natural habitats, model theory. Łoś's theorem — the central model-theoretic result about filters and ultrafilters — says that, if a set of propositions comprises an ultrafilter, some model will make true exactly the sentences expressing propositions in the filter. It is a powerful model-theoretic tool for showing that certain models exist. But if we don't understand the tool, we can't use it.

What philosophically familiar ideas are we going to use here? Those of *possible worlds* and *propositions*. In particular, we can think of ultrafilters as a special kind of set of propositions, or a special kind of 'propositional theory'. A filter is a set of propositions that meet certain conditions, and an ultrafilter is a filter that meets an additional condition. In particular, a filter is a set of propositions that is closed under (finite) conjunction and implication; and an ultrafilter is a filter that is also negation-complete.

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<sup>1</sup>They go unmentioned, for instance, in popular textbooks such as Boolos et al. 2002, Gamut 1991a,b, and Sider 2010. Enderton 2001: 142 mentions them briefly only to set them aside. Discussions in other books popular in philosophy graduate programs (e.g. Mendelson 1997: 129–136) aren't much more accessible than those in standard mathematics texts, such as Chang and Keisler 1990: 211–219.

# 1 PROPOSITIONAL THEORIES

## 1.1 Background

Suppose we have a set of worlds  $W$ . They may be possible worlds, but they don't have to be. The only things that matters about them (for our purposes) are that (i) they make sentences true or false, and (ii) every world is 'classically logical' — that is, for any world  $w$ , there is a model  $\mathcal{M}$  that agrees with  $w$ . More precisely, for every world  $w$ , there is a model  $\mathcal{M}_w$  where  $\phi$  is true at  $w$  if and only if  $\phi$  is true on  $\mathcal{M}_w$ . (We will make this even more precise later, but this is enough to be getting on with for now.)

With this in hand, here are some definitions.

### Definition 2.

- A  **$W$ -proposition** is any subset of  $W$ .
- If  $p$  and  $q$  are  $W$ -propositions, their **conjunction** is  $p \cap q$ .
- If  $p$  is a  $W$ -proposition, its **negation** is  $W - p$ , that is, the set of worlds in  $W$  that are not in  $p$ .
- If  $p$  and  $q$  are  $W$ -propositions, then  **$p$  implies  $q$**  iff  $p \subseteq q$ .
- $\emptyset$  is the **impossible proposition**.
- $W$  is the  **$W$ -necessary** proposition.
- If  $\phi$  is a sentence, the  **$W$ -intension of  $\phi$** , written  $\llbracket \phi \rrbracket_W$ , is the set of worlds where  $\phi$  is true.

When we know what the set  $W$  is, we drop all the ' $W$ 's in the above and just call them propositions, intensions, and so on. We also write  $W - p$  as ' $-p$ ', and write  $p \subseteq q$  as ' $p \Rightarrow q$ '. That the definitions for propositions are pretty good is shown by the fact that — given our condition (ii) on  $W$  — we have:

### Proposition 3.

- $\llbracket \top \rrbracket = W$ .
- $\llbracket \perp \rrbracket = \emptyset$ .
- $\llbracket \sim \phi \rrbracket = -\llbracket \phi \rrbracket$ .
- $\llbracket \phi \wedge \psi \rrbracket = \llbracket \phi \rrbracket \cap \llbracket \psi \rrbracket$ .
- $\llbracket \phi \rrbracket \Rightarrow \llbracket \psi \rrbracket$  iff  $\llbracket \phi \rightarrow \psi \rrbracket = \llbracket \top \rrbracket$ .

## 1.2 Filters

### 1.2.1 As Finitely Consistent Theories

Just as we often call a set of sentences a ‘theory’, we can call a set of propositions a ‘theory’ as well. More precisely, if  $T$  is any set of  $W$ -propositions, we will call it a  **$W$ -propositional theory**. From this perspective, a filter is simply a special kind of propositional theory.

#### Definition 4.

If  $T$  is a  $W$ -propositional theory, then  $T$  is a **filter over  $W$**  (or just ‘ $W$ -filter’) iff it is non-empty and

- (i) it is closed under pairwise conjunction;
- (ii) it is closed under implication; and
- (iii) it does not contain the impossible proposition.

By working through our definitions of ‘implication’ and the like, you can see that this definition is the same as definition 1, with  $W$  replacing  $I$ . But philosophers tend to have a much better developed sense about how to think about propositions, conjunction, and implication than about arbitrary sets, powersets, and intersections, making our current definition easier to digest.

So a filter is a kind of propositional theory. Why should we care about it? Well, because filters are something like *consistent* propositional theories. Here’s the intuitive idea. Suppose a propositional theory contained contradictory premises. Then those premises have, as an implication, the impossible proposition. But since a filter is closed under conjunction and implication, if a filter had contradictory premises, it would contain the impossible proposition — which it doesn’t.

This intuitive idea is close, but not quite, right. Filters are closed under pairwise conjunction: if  $p$  and  $q$  are in an ultrafilter, so is  $p \cap q$ . As a result, they will also be closed under finite conjunction, since any finite conjunction can be gotten by a finite number of pairwise conjunctions. ( $p$ -and- $q$ -and- $r$  is just  $p$ -and-( $q$ -and- $r$ ), and so on.) But a filter could still, in principle, contain an *essentially infinite* contradiction. For instance, it could contain, for each  $n$ , the proposition that there are at least  $n$   $F$ ’s, but then also include the proposition that there are only finitely many  $F$ ’s. No finite conjunction of those propositions are inconsistent; but of course the conjunction of all infinitely many of them is.

This is a big part of why filters are interesting. Closure under finite conjunction is much easier to achieve than closure under arbitrary conjunction. If we can get a propositional theory that has the weaker property, we can use various results about filters to learn more about it. Ultrafilters can, for instance, be used to show the first-order *compactness theorem*. This theorem says that if every finite subset of a set of formulas  $\Gamma$  has a model, then so does  $\Gamma$ . In other words, when it comes to first-order logic, any set of formulas that is finitely consistent is also infinitely consistent. The reason we can prove this is that we can use the finite consistency

of all of  $\Gamma$ 's subsets to come up with a finitely consistent propositional theory, and then use facts about that propositional theory to construct a model that makes *all* of  $\Gamma$  true.

### 1.2.2 Alternative Definitions

Sometimes textbooks provide alternative definitions of filters. A reasonably common variant drops the third condition from the definition — the one that excludes the impossible proposition — and goes on to distinguish ‘proper filters’, which are the  $\emptyset$ -free ones, from ‘improper filters’.<sup>2</sup> This is mere terminological deviation. If we wanted, we could call propositional theories closed under implication and conjunction ‘prefilters’. Then our ‘prefilters’ would correspond to their ‘filters’ and our ‘filters’ would correspond to their ‘proper filters’. (Their ‘improper filters’ would be our ‘prefilters that are not filters.’)

With a fixed background set of worlds  $W$ , a different definition for ‘filter’ (in our ‘proper’ sense) is also possible:

#### Definition 5.

If  $T$  is a  $W$ -propositional theory, then  $T$  is a **filter\*** over  $W$  iff

- (i\*) it is closed under pairwise conjunction;
- (ii\*) if a  $W$ -conjunction is in it, so are both  $W$ -conjuncts; and
- (iii\*) it contains  $W$  but not  $\emptyset$ .

It is not terribly difficult to see that the two definitions are equivalent.

#### Proof.

Note that conditions (i) and (i\*) are identical.

First, suppose  $T$  is a filter\* over  $W$ . We need to show it meets conditions (ii) and (iii). Condition (iii) is a conjunct of (iii\*). For (ii), note that if  $p \Rightarrow q$ , then  $p = p \cap q$ . So if  $p \in T$  and  $p \Rightarrow q$ , by (ii\*),  $q \in T$ . So  $T$  is closed under implication, satisfying (ii).

For the other direction, suppose  $T$  is a filter. Then it is non-empty, so by condition (ii) it contains  $W$ ; and by (iii) it does not contain  $\emptyset$ , so (iii\*) is satisfied. For (ii\*), note that a conjunction implies each of its conjuncts, so closure under implication tells us that if  $p \cap q \in T$ , both  $p$  and  $q$  are, too. ⊢

## 2 ULTRAFILTERS

So filters can be thought of as propositional counterparts of (finitely) consistent theories. But sometimes we are interested in more than just consistent theories — we want theories that are consistent and *complete*, ones that rule on every question

<sup>2</sup>E.g. Chang and Keisler 1990: 221–222.

we can ask of them. More precisely, if  $\Gamma$  is a theory of a language  $\mathcal{L}$ , it is **negation-complete** if and only if, for any sentence  $\phi$  of  $\mathcal{L}$ , either  $\phi$  or  $\sim\phi$  is in  $\Gamma$ . (Of course, if it is both consistent and complete, then exactly one of  $\phi$  and  $\sim\phi$  will be in  $\Gamma$ .)

What goes for linguistic theories goes for propositional ones, too. Filters are the (finitely) consistent ones. But a filter is an *ultrafilter* iff it is negation-complete. More precisely, if  $W$  is a set of worlds and  $T$  a  $W$ -filter, then  $T$  is **ultra** iff for any  $W$ -proposition  $p$ , either  $p$  or  $\neg p$  is in  $T$ .

**Lindenbaum's lemma** is an important theorem of metalogic which says that any consistent theory of a language  $\mathcal{L}$  can be extended to one that is both consistent and negation-complete. Here's the basic idea. We start with a consistent theory  $\Gamma$ ; think of it as a bucket filled with sentences of  $\mathcal{L}$ . Take an enumeration of all the sentences of  $\mathcal{L}$ :  $\phi_1, \phi_2, \dots$ . Then we go through the list, one by one: if we can throw  $\phi_i$  into the bucket without creating inconsistency, we do so. Otherwise, we keep it out. Once all the sentences have been gone through, for any  $\phi$ , we will have considered both  $\phi$  and  $\sim\phi$  when their respective turns came around, and thrown exactly one into the bucket. The theory consisting of everything at the end of the bucket will thus be both consistent and negation-complete.

We have a similar result for filters and ultrafilters, and the proofs are similar. Here, we start with a filter  $T$  over a set of worlds  $W$ , thinking of it as our starter bucket. Then we take a well-ordering of the  $W$ -propositions,<sup>3</sup> and run through those one by one, throwing them into the bucket if the result will remain consistent, and keeping them out otherwise.

Here we have to be careful, though. If we take a  $W$ -filter and expand it just by inserting a new  $W$ -proposition, the result probably won't be a filter. Filters, after all, are supposed to be closed under implication and pairwise conjunction. What we need, rather, is something like 'the result of adding  $p$  to a filter  $F$  and then closing up the result under conjunction and implication'. In this case, the proof will proceed by adding, *in this sense*, a proposition to the bucket exactly when the result would still be consistent. We'll make all this precise below, including the relevant notion of 'adding' a proposition to a filter.

## 2.1 Additions

Suppose we have a  $W$ -filter  $T$  and a  $W$ -proposition  $p$ . Take the set  $T \cup \{p\}$ . This generally won't be closed under conjunction or implication. Suppose that  $a \in T$  and  $p \subseteq b \subseteq W$ ; there's no reason to think that  $a \cap p$  or  $b$  will be in  $T \cup \{p\}$ .

But that's no obstacle; after adding  $p$  to  $T$ , we add in the missing ones as well. Here's one we could do it. First, we conjoin  $p$  with each  $W$ -proposition in  $T$ . Then we add in any other propositions that these imply. The second step makes the result closed under implication, and since  $T$  was already conjunction-closed, the first step makes our new theory conjunction-closed, too. Furthermore, since  $p \cap q$  always implies  $q$ , every proposition in  $T$  will be in our new theory as well.

We can do this in two steps, and with a pair of definitions.

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<sup>3</sup>This requires the axiom of choice; more on this below.

**Definition 6.**

Let  $S$  be any  $W$ -theory. (It needn't be a filter.)

**Conjunction Closure of  $S$ :**  $S^\cap = \{p \cap q \subseteq W : p, q \in S\}$ .

**Implication Closure of  $S$ :**  $S^\Rightarrow = \{p \subseteq W : q \Rightarrow p \text{ for some } q \in S\}$ .

**Addition:**  $S + p = (S \cup \{p\})^{\cap\Rightarrow}$ .

(Textbooks will often define the third instead as

$$S + p = \{q \subseteq W : \text{for some } a \in S, a \cap p \Rightarrow q\}.$$

Readers can work through the set theoretic definitions to confirm that the two definitions are equivalent.)

It is clear that  $S^\cap$  will be closed under conjunction, and  $S^\Rightarrow$  will be closed under implication. But will  $S^{\cap\Rightarrow}$  be closed under conjunction? By adding in all the things that are implicated by members of  $S^\cap$ , will we have lost closure under conjunction? Not according to the next proposition.

Neither of these will happen, though. First we have:

**Proposition 7.**

If  $S$  is a  $W$ -theory (not necessarily a filter) and closed under conjunction, then  $S^\Rightarrow$  is also closed under conjunction. Furthermore, if  $\emptyset \notin S$ , then  $\emptyset \notin S^\Rightarrow$ .

**Proof.**

Suppose  $p, q \in S^\Rightarrow$ . Then for some  $p^s, q^s \in S$ ,  $p^s \Rightarrow p$  and  $q^s \Rightarrow q$ . But  $S$  is closed under conjunction, so  $p^s \cap q^s \in S$ , and  $p^s \cap q^s \Rightarrow p \cap q$ , so  $p \cap q \in S^\Rightarrow$ .

For the second part, if  $\emptyset \in S^\Rightarrow$ , then for some  $q \in S$ ,  $q \Rightarrow \emptyset$ , which can only happen if  $\emptyset = q$ . ⊥

As a corollary, if  $S$  doesn't contain the impossible proposition, then  $S^\Rightarrow$  will be a filter; it is called the **filter based on  $S$** .

So if we have a  $W$ -filter  $T$ , we can add a  $W$ -proposition  $p$  to it and close the remainder up under conjunction and implication. What can we say about this set? Well, for one thing, it will be an extension of  $T$  — it will include everything in  $T$ , and (unless  $p$  was already in it) more besides.

But *will it be a filter?*

Yes — *unless*  $\neg p$  was already in  $T$ . We know that  $T$  is a filter, and that  $T + p$  is closed under conjunction and implication (thanks to proposition 7.) So it will only fail to be a filter if it includes the impossible proposition. Given what 7 shows, it will only do that if, when we close it under conjunction, we add the impossible proposition, and that will only happen if  $\neg p$  was already in  $T$ .

Here are the facts about additions, rolled together into a tidy lemma.

**Lemma 8.**

Suppose  $T$  is a filter over  $W$  and  $p$  is a  $W$ -proposition. Then

- (i)  $T \subseteq T + p$  and  $p \in T + p$ .
- (ii)  $T + p$  is closed under conjunction and implication.
- (iii)  $T + p$  is a filter iff  $-p \notin T$ .

**Proof.**

For (i): Since  $T + p = (T \cup p)^{\cap \Rightarrow}$ , it's enough to show that each operation creates a superset. If  $q = q \cap q$ , so if  $q \in S$ ,  $q \in S^\cap$ . And  $q \Rightarrow q$  so if  $q \in S$ ,  $q \in S^\Rightarrow$ .

For (ii), it is clear that  $(T \cup p)^\cap$  is closed under conjunction, so  $T + p$  is by proposition 7. And it is equally clear that  $(T \cup p)^{\cap \Rightarrow}$  is closed under implication.

For (iii): If  $-p \in T$ , then by part (i),  $-p \in T + p$  and  $p \in T + p$ . by part (ii),  $\emptyset = -p \cap p \in T + p$ , making it not a filter.

Conversely, suppose that  $T + p$  is not a filter. By part (ii), this can only happen if  $\emptyset \in T + p$ . By proposition 7,  $\emptyset \in (T \cup \{p\})^\cap$ . So for some  $a \in T$ ,  $a \cap p \Rightarrow \emptyset$ . So  $a \Rightarrow -p$ ; since  $T$  is closed under implication, it contains  $-p$ .  $\dashv$

## 2.2 Three Concepts

A filter is ultra when it is negation-complete. But this isn't the only property it might have. For a  $W$ -filter  $T$ , consider:

**Definition 9.**

**Maximality:**  $T$  is the only filter over  $W$  that includes  $T$ .

That is, if  $T \subseteq T'$  and  $T'$  is a filter over  $W$ , then  $T = T'$ .

**Prime:**  $T$  is disjunction-complete.

That is, for any  $W$ -propositions  $p$  and  $q$ ,  $p \cup q \in T$  iff either  $p \in T$  or  $q \in T$ .

**Ultra:**  $T$  is negation-complete.

That is, for any  $W$ -proposition  $p$ , either  $p$  or its negation is in  $T$ .

Conceptually, these are all different. Maximality tells us that  $T$  is a 'biggest' filter: there's no way to add propositions to the filter without making it a non-filter. Primeness tells us that it never contains a disjunction without also containing a disjunct. And we have already seen that ultra-ness tells us that it answers every question we can put to it.

Despite the conceptual differences, any filter that has one has all three. That is,

**Theorem 1.**

If  $T$  is a  $W$ -filter, then it is maximal iff it is prime iff it is ultra.

We'll prove it here, leaving the 'W's off for readability.

**Proof.**

Suppose first that  $T$  is an ultrafilter. Take any proposition  $p$ . Since  $T$  is ultra, either  $p$  or  $\neg p$  is in  $T$ . Suppose we extend  $T$  by adding  $p$ . If  $p \in T$ , then our addition doesn't extend it. If instead  $\neg p \in T$ , then if our extension is closed under conjunctions,  $p \cap \neg p = \emptyset$  will be in this extension and so it will not be a filter. Thus it is maximal.

Next, suppose that  $T$  is maximal. If  $p$  is in  $T$ , then so is  $p \cup q$ , since  $p$  implies  $p \cup q$  and filters are closed under implication. Conversely, suppose  $p \cup q$  is in  $T$  but neither  $p$  nor  $q$  are. Since  $T$  is maximal,  $\neg p \in T$  and  $\neg q \in T$  by lemma 8. Then  $\neg p \cap \neg q = \neg(p \cup q) \in T$ , so  $T$  is also not a filter. Thus one of  $p$  or  $q$  is in  $T$ .

Finally, suppose  $T$  is prime. Since  $W$  is in  $T$  (being implied by every proposition), and for any proposition  $p$ ,  $p \cup \neg p = W$ , by  $T$ 's primeness, either  $p \in T$  or  $\neg p \in T$ . So  $T$  is ultra. ◻

The proof shows we can go in a circle from ultra-ness, through maximality and primeness, and back to ultra-ness, which means all three properties are equivalent.

### 2.3 The Central Theorem on Ultrafilters

The propositional version of Lindenbaum's lemma is

**Theorem 2. The Central Theorem on Ultrafilters**

Every filter can be extended to an ultrafilter.

We can prove it along the lines of the 'bucket' proof sketched above, where we 'throw'  $p$  into the bucket  $T$  by trading the latter for  $T + p$ . Before we do, though, a few points are in order.

First, we can't of course literally go through an enumeration of  $W$ -propositions (or sentences) and make an active decision whether or not to throw them in. But we don't have to. Our bucket-talk represents a set theoretic construction. We start with a set  $T_0$ , which is the initial stage of our bucket, and then define sets  $T_1, T_2, \dots, T_i, \dots$  that represent later bucket-stages. The final theory is then simply the union of all the  $T_i$ 's — of all the 'bucket stages', as it were.

Second, we generally assume that languages are finite, in which case there will only be countably many sentences of a given language. As a result, there's no issue ordering the sentences when we prove Lindenbaum's lemma. However, so long as there are infinitely many worlds, there will be *uncountably* many  $W$ -propositions. In that case, they aren't enumerable and we can't assume any ordering of them.

The *well-ordering theorem* says that any set can be well-ordered. It is equivalent to the axiom of choice. So if we assume the axiom of choice, we can use it to

well-order the  $W$ -propositions. Then we continue with our ‘throw-it-in-the-bucket’ strategy, this time using the well-ordering of  $W$ -propositions. Each  $W$ -proposition gets assigned an ordinal number, and we can then go through the ordinals to construct our ultrafilter.<sup>4</sup>

Natural numbers come in two types: zero, and the ‘successor’, or ‘plus one’ of some natural number. Ordinal numbers come in *three* types: zero, successors, and *limit ordinals*, which are the unions of all the ordinals smaller than them. As a result, our proof will need to use transfinite induction, which has a base case, a case for successor ordinals, and a case for limit ordinals.

With that out of the way, let’s get to proving.

**Proof.**

Let  $T$  be a  $W$ -filter and take an indexing of the  $W$ -propositions by ordinals. When  $p$  is indexed by  $\alpha$  we call it  $p_\alpha$ . Thanks to the fact that ordinals are well-ordered, there will be a least ordinal  $\gamma$  where every  $W$ -proposition has an index less than  $\gamma$ . Now we define a series of filters as follows.

$$T_0 = T$$

$$T_{\alpha+1} = \begin{cases} T_\alpha + p_\alpha, & \text{if that is a filter, and} \\ T_\alpha & \text{otherwise.} \end{cases}$$

$$T_\alpha = \bigcup_{\beta < \alpha} T_\beta \text{ for limit } \alpha$$

We want to show that each  $T_\alpha$  in this series is a filter, and that if  $\beta < \alpha$ , then  $T_\beta \subseteq T_\alpha$ . This is done by induction on the ordinals. In the base case  $T_\alpha = T_0$ , which is a filter by assumption. For the induction step, we assume that if  $\beta < \alpha$ , then  $T_\beta$  is a filter.

When  $\alpha$  is a successor ordinal, it is a filter by construction, and  $T_\alpha \subseteq T_{\alpha+1}$  by lemma 8.i.

For a limit ordinal  $\alpha$ , it’s clear that if  $\beta < \alpha$ , then  $T_\beta \subseteq T_\alpha$ . But we need to show that  $T_\alpha$  is a filter. To do that we have to show that it meets all three conditions of being a filter. For the third condition, if  $\emptyset \in T_\alpha$ , then  $\emptyset$  would have to be a member of some  $T_\beta$  for  $\beta < \alpha$ , which contradicts the induction hypothesis that all such  $T_\beta$  are filters.

For implication closure, suppose that  $q \in T_\alpha$  and  $q$  implies  $r$ . To be in  $T_\alpha$ ,  $q$  must be in some  $T_\beta$  for  $\beta < \alpha$ . But since  $T_\beta$  is a filter, it is closed under implication, so  $r \in T_\beta$  and thus also in  $T_\alpha$ .

For conjunction closure, suppose that  $q$  and  $r$  are in  $T_\alpha$ . Then there are  $\beta_q$  and  $\beta_r$ , both less than  $\alpha$ , where  $q \in T_{\beta_q}$  and  $r \in T_{\beta_r}$ . If  $\beta$  is the larger of the two, then  $q, r \in T_\beta$ . But  $T_\beta$  is a filter, and so closed under conjunction, so  $q \cap r \in T_\beta$ . Since  $T_\alpha$  includes  $T_\beta$ ,  $q \cap r$  is in  $T_\alpha$  also.

So every  $T_\alpha$  in our construction is a filter. Now we take  $T_\gamma$ , the union of all the  $T_\alpha$ ’s in our construction. It is also a filter, by the above argument. We just need to

<sup>4</sup>The Central Theorem on Ultrafilters (or the Prime Ideal Theorem, with which it is equivalent) is not a theorem of regular, choice-free Zermelo-Frankel set theory, but it is also weaker than that plus choice. So instead of assuming choice we could just assume the theorem itself. But that isn’t nearly as illuminating.

show that it is an ultrafilter. To do this, note that if  $p$  is any  $W$ -proposition it is  $p_\alpha$  for some  $\alpha$ . If  $T_\alpha + p_\alpha$  was a filter, then  $p \in T_{\alpha+1}$  and so in  $T_\gamma$ . If instead  $T_\alpha + p_\alpha$  was not a filter, then by lemma 8.iii,  $\neg p$  was already in  $T_\alpha$ , so  $\neg p \in T_\gamma$ .  $\dashv$

### 3 FILTERS, ULTRAFILTERS, AND FINITUDE

#### 3.1 The Consequences of a Proposition

Consider some proposition  $q$  — the proposition that snow is white, perhaps — and think of all the propositions that it entails. These will form a filter, because if *snow is white* entails two propositions it entails their conjunction, entailment is transitive, and *snow is white* is possible.

More generally, for any  $W$ -proposition  $p$ , let the **consequences** of  $p$  be the set

$$C(p) := \{q \subseteq W : p \Rightarrow q\}.$$

This will be closed under conjunction and implication, and if  $p$  is not the impossible proposition,  $C(p)$  will be a filter. In this case,  $C(p)$  is called the **principal ultrafilter generated by  $p$** .

Some filters will be of this sort — they will be  $C(p)$  for some  $p$ . These are called **principal filters**. Here's an interesting fact about them.

#### Proposition 10.

If a filter  $T$  contains any finite proposition — that is, any proposition containing only finitely many worlds — then it is principal.

#### Proof.

Suppose  $T$  contains some finite proposition  $p$ . Then there will be some smallest subset  $q$  of  $p$  that is also in  $T$ . (It might be  $p$  itself.) Notice, however, that for any proposition  $r$  in  $T$ ,  $r \cap q$  can't be empty, because  $T$  is a filter. Furthermore,  $r \cap q$  will be in the filter. But  $r \cap q \Rightarrow q$ , so if  $q$  really is the smallest proposition in  $T$ ,  $r \cap q = q$ , which means  $q \Rightarrow r$ . Since  $r$  was arbitrary,  $T = C(q)$ .  $\dashv$

When  $T$  isn't just a filter, but an *ultrafilter*, something even stronger holds.

#### Proposition 11.

If  $T$  is an ultrafilter containing any finite proposition, there is some single world  $w$  where  $T = C(\{w\})$ .

To see why, recall that an ultrafilter is also prime, and that a finite proposition is the (finite) disjunction of all the worlds in it. So we can check each world one by one; either its singleton will be in  $T$ , or we'll keep on going down the list until we get to the last one.

Note that the converse of proposition 10 doesn't hold. If  $p$  is a proposition true at (containing) infinitely many worlds,  $C(p)$  is still a principal ultrafilter, but it won't contain any proposition 'smaller' than  $p$ , so it won't contain any finite propositions.

Note also how the proof breaks down when  $T$  doesn't contain any finite propositions: If a proposition  $p$  has infinitely many worlds in it, there's no guarantee that  $T$  will contain a *smallest* subset of it. This suggests that maybe there are *non-principal* filters, which (by proposition 10) will only have infinite propositions in them.

This is, in fact, the case. Suppose  $W$  has infinitely many worlds in it. In this case, call a  $W$ -proposition  $p$  **rarely false** if it is false at only finitely many worlds — equivalently, if  $\neg p$  is finite. Then the set of all rarely false propositions is a filter.

To see why, note first that if  $p$  is rarely false and  $p \Rightarrow q$ , then  $\neg p$  is finite and  $\neg q \Rightarrow \neg p$ . But remember that ' $\Rightarrow$ ' is just ' $\subseteq$ ', and the subset of a finite set must be finite. So  $\neg q$  is finite, meaning  $q$  is rarely false, too.

Second, if  $\neg p$  and  $\neg q$  are finite, so is  $\neg p \cup \neg q$ . But  $\neg p \cup \neg q = \neg(p \cap q)$ . So if  $p$  and  $q$  are rarely false,  $\neg(p \cap q)$  is finite and so  $p \cap q$  is rarely false, too. So the rarely false propositions are closed under conjunction.

Finally, of course,  $\emptyset$  isn't rarely false. Quite the opposite, in fact. (And  $W$  itself is rarely false, because it is false at zero worlds and zero is a finite number.)

So the rarely false propositions form a filter. But note that it can't be principal. If it were, then it would be  $C(p)$  for some rarely false proposition  $p$ . But if  $p$  is a rarely false proposition, by just removing one world from it, we get another rarely false proposition that  $p$  doesn't imply. So the rarely false propositions form a non-principal ultrafilter. In fact, it has a name — it is called the **cofinite filter**. More precisely, for any set  $W$ , the cofinite filter on  $W$  consists of all the subsets of  $W$  that are missing only finitely members of  $W$ .

### 3.2 Finite and Infinite Conjunction Closure

We can think of proposition 10 as being related to the fact that filters are closed under (finite) conjunction. In general, whenever we have a principal filter  $C(p)$ , the proposition  $p$  will be the conjunction of everything in  $C(p)$ . What proposition 10 tells us is that, when a filter  $T$  has a finite proposition in it, it also contains the conjunction of everything in  $T$ . That's because — very roughly — when a filter contains a finite proposition, the conjunction of everything in the filter is equivalent to the conjunction of finitely many things in it.

On the other hand, the conjunction of everything in  $W$ 's cofinite filter  $F$  is just the empty set. That's because, for any world  $w$ , there is some proposition in  $F$  that doesn't contain  $w$ . So there is *no* world in every one of  $F$ 's propositions, meaning the intersection (i.e., conjunction) of all infinitely many of them together must be empty.

Here's one place where the distinction between 'closed under pairwise conjunction' and 'closed under arbitrary conjunction' matters. Any theory closed under arbitrary conjunction will consist of exactly the consequences of that conjunction.

If filters were closed under *that*, they would all be principal. It's the fact that they aren't which allows us to have non-principal ultrafilters.

### 3.3 Principal and Non-Principal Ultrafilters

So, are there non-principal ultrafilters? As with the Central Theorem on Ultrafilters, Zermelo-Frankel set theory doesn't settle this on its own. However, the Central Theorem (and thus anything that implies it, such as the axiom of choice) *does* settle the matter with an answer in the affirmative.

Recall the cofinite filter from above — the filter containing all of the rarely-false propositions. It won't be an ultrafilter; if we partition the set of worlds  $W$  into two infinite sets,  $p$  and  $\neg p$ , neither of these will be rarely false, and so won't be in the cofinite filter. However, according to the Central Theorem, there *will* be an ultrafilter that contains the cofinite filter (and so contains one of  $p$  and  $\neg p$ ).

Let  $T$  be such an ultrafilter. Could  $T$  be principal? If it were, then there would be some proposition  $p$  that implies everything in  $T$ . But this can't happen. Take some world  $w$  that  $p$  is true at — that is, a world  $w \in p$ . If we take any rarely-false proposition  $q$  and remove  $w$  from it, the resulting proposition is also rarely false. (It is false, at *most*, at one more world than  $q$  is false at.) But  $p$  then can't imply this proposition in  $T$ , so  $T$  can't be the principal ultrafilter for  $p$ . Since this holds for *any*  $p \in T$ ,  $T$  must be non-principal.

## 4 INTENSIONAL SPACES

So far we've focused on the boolean side of things — the way that ultrafilters relate to the 'and'-ing and 'or'-ing of propositions (or properties). But logic includes the logic of quantification, and ultrafilter-based tools extend to this as well.

We assumed, at the outset, that every world had a model which agreed with it, but that assumption has remained largely dormant thus far. That's all about to change; we need the quantificational structure of models to connect propositions with quantifiers.

### 4.1 The Model-Theoretic Background

Let's fix some ideas about model theory. Suppose we have a first-order language  $\mathcal{L}$ , with a fixed stock of names and predicates, infinitely many variables, and primitive logical symbols ' $\sim$ ', ' $\wedge$ ', ' $\exists$ ', and ' $=$ '. Other logical symbols are to be defined in the usual way.

A **model**  $\mathcal{M}$  is an ordered pair  $\langle D, I \rangle$  of a non-empty domain  $D$  and an interpretation function  $I$ . For any constant  $\alpha$  in  $\mathcal{L}$ ,  $I(\alpha) \in D$ , and for any  $n$ -adic predicate  $\Pi$  in  $\mathcal{L}$ ,  $I(\Pi) \subseteq D^n$ . (That is,  $I(\Pi)$  is a set of  $n$ -tuples of things drawn from  $D$ .)

For each model  $\mathcal{M}$  of  $\mathcal{L}$ , a **variable assignment over  $\mathcal{M}$**  is a function from variables of  $\mathcal{L}$  to elements of  $D$ . A **term** is any constant or variable, and the

**denotation** of a term  $\alpha$  on a model  $\mathcal{M}$  relative to a variable assignment  $a$ , written  $\alpha^{\mathcal{M},a}$ , is  $I(\alpha)$  if a constant and  $a(\alpha)$  if a variable. If  $a$  is a variable assignment over  $\mathcal{M}$  and  $o \in D$ ,  $a[o \triangleright x]$  is the assignment just like  $a$  except that it assigns the object  $o$  to the variable  $x$ .

To avoid clutter, we write a sequence  $\langle \alpha_1, \dots, \alpha_n \rangle$  as  $\vec{\alpha}$ . Now we can recursively define truth of an open formula on a model relative to a variable assignment, written  $\mathcal{M}, a \models \phi$ . The definition runs:

**Definition 12.**

Let  $\mathcal{M}$  be a model and  $a$  a variable assignment. Then

- (i)  $\mathcal{M}, a \models \Pi \vec{\alpha}$  iff  $\langle \overrightarrow{\alpha^{\mathcal{M},a}} \rangle \in I(\Pi)$ .
- (ii)  $\mathcal{M}, a \models \alpha = \beta$  iff  $\alpha^{\mathcal{M},a} = \beta^{\mathcal{M},a}$ .
- (iii)  $\mathcal{M}, a \models \sim\phi$  iff  $\mathcal{M}, a \not\models \phi$ .
- (iv)  $\mathcal{M}, a \models \phi \wedge \psi$  iff  $\mathcal{M}, a \models \phi$  and  $\mathcal{M}, a \models \psi$ .
- (v)  $\mathcal{M}, a \models \exists x\phi$  iff for some  $o \in D$ ,  $\mathcal{M}, a[o \triangleright x] \models \phi$ .

When  $\phi$  is a formula with  $x$  free, we allow ourselves to write it as  $\phi(x)$ , with  $\phi(\alpha)$  being the result of replacing all those (unbound)  $x$ 's with  $\alpha$ . Finally, we say that a formula is true on  $\mathcal{M}$ , or  $\mathcal{M} \models \phi$ , when it is true on  $\mathcal{M}$  relative to every variable assignment over  $\mathcal{M}$ ; and  $\mathcal{M} \models \Gamma$  if  $\mathcal{M} \models \phi$  for every  $\phi \in \Gamma$ . Finally,  $\Gamma \models \phi$  iff, for every  $\mathcal{M}$ , if  $\mathcal{M} \models \Gamma$ , then  $\mathcal{M} \models \phi$ .

It is routine to show that if a formula with no free variables is true on  $\mathcal{M}$  relative to any variable assignment then it is true on  $\mathcal{M}$  relative to all of them.

## 4.2 Models and Intensions

Models are *extensional*: they assign referents to names and extensions to predicates. But when dealing with possible worlds semantics, we are interested in *intensional* assignments. These are often taken as functions from possible worlds to extensions. For instance, the extension of 'dog' is the set of all the dogs; the intension of 'dog' is the function from each possible world  $w$  to the set of things that are dogs in  $w$ .

Our 'possible worlds' don't have things in them — the set  $W$  is really just an index set — but if every world in  $W$  is associated with a model, we can use things in the model instead. For instance, if  $w$  is a world and  $\mathcal{M}^w$  is its associated model, then there is the set of things that are in the extension of 'dog' in  $\mathcal{M}^w$ . So, for a set of worlds  $W$ , if each one has an associated model, we can have something like an 'intension' defined on these worlds and models.

First, a definition to make this idea of 'an associated model' precise.

**Definition 13.**

An **intensional space** over a language  $\mathcal{L}$  is an ordered pair  $\mathcal{W} = \langle W, i \rangle$ ;  $W$  is a set of worlds and  $i$  a function where, for every  $w \in W$ ,  $i(w)$  is a model of  $\mathcal{L}$ .

When  $\mathcal{M} = i(w)$ , we streamline notation by writing it as ' $\mathcal{M}^w$ ', the ordered pair  $\langle D^w, I^w \rangle$ . Notice that the models in an intensional space are all models of the very same language.

Now we can define the notion of the intension of a name or predicate, relative to an intensional space.

**Definition 14.**

Let  $\mathcal{W}$  be an intensional space over  $\mathcal{L}$ ,  $\Pi$  a predicate of  $\mathcal{L}$ , and  $\alpha$  a name of  $\mathcal{L}$ . Then:

- The  **$\mathcal{W}$ -intension of  $\Pi$** , written ' $\llbracket \Pi \rrbracket_{\mathcal{W}}$ ', is the function from  $W$  defined by  $\llbracket \Pi \rrbracket_{\mathcal{W}}(w) = I^w(\Pi)$ .
- The  **$\mathcal{W}$ -intension of  $\alpha$** , written ' $\llbracket \alpha \rrbracket_{\mathcal{W}}$ ', is the function from  $W$  defined by  $\llbracket \alpha \rrbracket_{\mathcal{W}}(w) = I^w(\alpha)$ .

Notice something about the second definition. Kripke (1972) taught us that names are 'rigid designators': They pick out the same thing in every possible world. The  $\mathcal{W}$ -intensions as defined are just functions from worlds to individuals in the world's associated domain, similar to Carnap's (1956) 'individual concepts'. While there will be some rigid individual concepts — ones that return the same individual for every world — those will just be special cases. Individual concepts can be highly non-rigid.

That's okay, though. Whatever the merits of Kripke's lesson for metaphysics and semantics generally, they don't apply here. Our talk of possible worlds and individual concepts are heuristic devices for making some model-theoretic technology easier to think about; they aren't proposals about metaphysics or natural language semantics.

Notice also that we've only defined intensions for particular symbols. But the structure is perfectly general. There will be plenty of functions from worlds to extensions not associated, in an intensional space, with any particular name or predicate. We generalize as follows:

**Definition 15.**

If  $\mathcal{W}$  is an intensional space, then:

- An  **$n$ -adic  $\mathcal{W}$ -relation** (or ' $\mathcal{W}$ -property', when  $n = 1$ ) is a function  $f$  from  $W$  to  $(\bigcap D^w)^n$  where, for each  $v \in W$ ,  $f(v) \subseteq (D^v)^n$ . (That is:  $f(v)$  is a set of  $n$ -tuples drawn from  $\mathcal{M}^v$ 's domain.)
- A  **$\mathcal{W}$ -individual concept** is a function  $g$  from  $W$  to  $\bigcap D^w$  where, for any

$$v, g(v) \in D^v.$$

So the  $\mathcal{W}$ -intension of any predicate is a  $\mathcal{W}$ -property or relation, and the  $\mathcal{W}$ -intension of a name is a  $\mathcal{W}$ -individual concept, or ‘ $\mathcal{W}$ -concept’ for short.

### 4.3 Intensions and $\mathcal{W}$ -Assignments

Back in section 1.1 we helped ourselves to the notion of ‘the set of worlds at which  $\phi$  is true’, and wrote it as  $\llbracket \phi \rrbracket_W$ . But since different assignments of worlds to models will result in different sets of worlds at which  $\phi$  is true, we should have instead relativized that notion to an intensional space. We can fix that now.

#### Definition 16.

If  $\mathcal{W}$  is an intensional space over  $\mathcal{L}$  and  $\phi$  is any sentence of  $\mathcal{L}$ , then  $\llbracket \phi \rrbracket_{\mathcal{W}} = \{w \in W : \mathcal{M}^w \models \phi\}$ .

In other words:  $\llbracket \phi \rrbracket_{\mathcal{W}}$  is the set of worlds in  $\mathcal{W}$  whose  $\mathcal{W}$ -associated models make  $\phi$  true.

When it comes to models, we define truth-in-a-model by first defining truth in a model relative to a variable assignment. This makes it easy to work with, letting us treat the semantic properties of complex expressions (relative to an assignment) as determined by the semantic properties of their parts (relative to an assignment). The idea is that, while we can’t say whether a formula like ‘Philosopher( $x$ )’ is true or false *simpliciter*, we can say that it is true or false relative to some assignment to the variable ‘ $x$ ’, and this helps to define truth-in-a-model for quantified sentences.

We can do something similar with open formulas and propositions. Given an intensional space  $\mathcal{W}$ , we know what  $\mathcal{W}$ -proposition ‘Philosopher(jason)’ expresses: it’s the set of worlds  $w$  where  $\mathcal{M}^w$  makes that sentence true. But we can also ask, relative to some way of assigning an intension to the variable ‘ $x$ ’, what  $\mathcal{W}$ -proposition ‘Philosopher( $x$ )’ expresses.

The answer isn’t hard to come by. ‘Philosopher’ expresses a  $\mathcal{W}$ -property  $\llbracket \text{Philosopher} \rrbracket_{\mathcal{W}}$ . This picks out, in each world  $w$ , a set of things  $P_w$  that are philosophers there.<sup>5</sup> If  $g$  is a  $\mathcal{W}$ -concept, then it picks out an object  $g(w)$  for each world  $w$ . At some worlds  $w$ ,  $\llbracket \text{Philosopher} \rrbracket_{\mathcal{W}}$  and  $g$  will coincide: the object  $g$  picks out,  $g(w)$ , will be in the set  $P_w$  of  $w$ -philosophers. At other worlds  $v$ , they won’t coincide:  $g(v)$  won’t be one of the  $v$ -Philosophers — that is,  $g(v) \notin P_v$ . Intuitively, if ‘ $x$ ’ is assigned to  $g$ , then the proposition expressed, relative to that assignment, by ‘Philosopher( $x$ )’ should be true at exactly the worlds where  $g$  and  $\llbracket \text{Philosopher} \rrbracket_{\mathcal{W}}$  coincide — that is, the worlds where  $g(w) \in P_w$ . So, relative to that assignment, the proposition expressed by ‘Philosopher( $x$ )’ should be the set of worlds where  $g(w) \in P_w$ .

<sup>5</sup>Note that  $P_w$  is just easier-to-parse shorthand for  $\llbracket \text{Philosopher} \rrbracket_{\mathcal{W}}(w)$ .

To make this perfectly general we should first define an **intensional assignment** on  $\mathcal{W}$  — or, equivalently, a  **$\mathcal{W}$ -assignment** — as a function from variables to  $\mathcal{W}$ -concepts. If we feed it a variable, we get a concept  $g$  out. If we then go on to feed  $g$  a world, we get an object in that world's model's domain.

Since  $\mathcal{W}$ -assignments return a function for each variable, we can also think of them, if we want, as devices that take two inputs — a variable and a world  $w$  — to determine an object in  $\mathcal{M}^w$ 's domain. If we feed this device a variable, we get a  $\mathcal{W}$ -concept: a function from worlds to objects in models' domains. But what happens if we just feed it a world  $w$  instead? In that case, we get a function from variables to objects in  $\mathcal{M}^w$ 's domain — in other words, we get a variable assignment on  $\mathcal{M}^w$ . (See figure 1.)

More generally, if  $A$  is any  $\mathcal{W}$ -assignment and  $w$  a world in  $\mathcal{W}$ , we define a function  $A^w$  by

$$A^w(x) = A(x)(w).$$

This will be the variable assignment on  $\mathcal{M}^w$  that takes each  $\mathcal{W}$ -concept  $g$  that  $A$  assigns to some variable and returns the object that  $g$  picks out in  $\mathcal{M}^w$ .

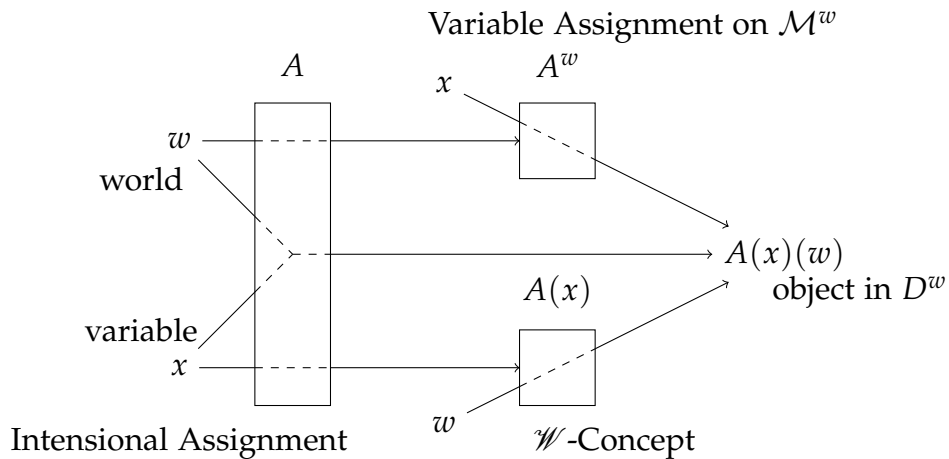


Figure 1: Intensional Assignments

With this in mind, we can state more directly what it is for a sentence to express, on an intensional space and relative to an intensional assignment, a particular  $\mathcal{W}$ -proposition.

**Definition 17.**

A formula  $\phi$   **$\mathcal{W}$ -expresses  $p$  relative to  $A$**  if and only if  $p$  is the set of worlds  $w$  where  $\mathcal{M}^w, A^w \models \phi$ . In symbols:

$$[[\phi]]_{\mathcal{W}, A} = \{w \in W : \mathcal{M}^w, A^w \models \phi\}.$$

With this machinery in place, we can extend proposition 3 to

**Proposition 18.**

If  $A$  is a  $\mathscr{W}$ -assignment:

- (i)  $\llbracket \top \rrbracket_A = W$ .
- (ii)  $\llbracket \perp \rrbracket_A = \emptyset$ .
- (iii)  $\llbracket \sim\phi \rrbracket_A = -\llbracket \phi \rrbracket_A$ .
- (iv)  $\llbracket \phi \wedge \psi \rrbracket_A = \llbracket \phi \rrbracket_A \cap \llbracket \psi \rrbracket_A$ .
- (v)  $\llbracket \phi \rrbracket_A \Rightarrow \llbracket \psi \rrbracket_A$  iff  $\llbracket \phi \rightarrow \psi \rrbracket_A = \llbracket \top \rrbracket_A$ .

I won't prove all of these here, but I'll talk through one of them by way of illustration. To show that  $\llbracket \sim\phi \rrbracket_A = -\llbracket \phi \rrbracket_A$ , we need to show that for any world  $w$  in  $\mathscr{W}$ ,  $w \in \llbracket \sim\phi \rrbracket_A$  if and only if it is not in  $\llbracket \phi \rrbracket_A$ . So we let  $w$  be an arbitrary world. If it is in  $\llbracket \sim\phi \rrbracket_A$ , then  $\mathcal{M}^w, A^w \models \sim\phi$ , which means  $\mathcal{M}^w, A^w \not\models \phi$ . But this means that  $w \notin \llbracket \phi \rrbracket_A$ . When  $w$  is not in  $\llbracket \sim\phi \rrbracket_A$ , a mirror-image of that argument tells us that  $w$  is in  $\llbracket \phi \rrbracket_A$ . The other instances of our expanded version of proposition 3 are shown in analogous ways.

Another result, related to proposition 18.v, also holds.

**Proposition 19.**

If  $\phi \models \psi$ , then for any  $\mathscr{W}$ -assignment  $A$ ,  $\llbracket \phi \rrbracket_A \Rightarrow \llbracket \psi \rrbracket_A$ .

If  $w$  is a world in  $\llbracket \phi \rrbracket_A$ , then  $\mathcal{M}^w, A^w \models \phi$ . But if  $\phi \models \psi$ , this means that  $\mathcal{M}^w, A^w \models \psi$ , too, so  $w \in \llbracket \psi \rrbracket_A$ . Proposition 19 will be very useful in streamlining proofs to come.

Notice, in passing, that proposition 19's converse isn't in general true. It may turn out that there are countermodels to the argument from  $\phi$  to  $\psi$ , but none of the worlds in  $\mathscr{W}$  get paired with any of them. In that case,  $\llbracket \phi \rrbracket_A$  might  $\mathscr{W}$ -imply  $\llbracket \psi \rrbracket_A$  even though  $\phi \not\models \psi$ .

#### 4.4 Shifting Variable Assignments

If  $a$  is a variable assignment on a model,  $a[o \triangleright x]$  is the one just like it except that it assigns  $o$  to the variable  $x$ . Given a  $\mathscr{W}$ -assignment  $A$  and a  $\mathscr{W}$ -concept  $g$ , we also define  $A[g \triangleright x]$  by

$$A[g \triangleright x](y, w) = \begin{cases} A(y, w), & \text{if } x \text{ and } y \text{ are different variables, and} \\ g(w) & \text{otherwise.} \end{cases}$$

So  $A[g \triangleright x]$  is just another intensional assignment, and we can recover both a variable assignment (by feeding it a world) or a  $\mathscr{W}$ -concept (by feeding it a variable). One very important fact relating this concept assignment and the variable assignments we get out of it is

$$A[g \triangleright x]^w = A^w[g(w) \triangleright x].$$

We can use this to prove one further fact about  $\mathscr{W}$ -propositions.

**Proposition 20.**

If  $A$  is a  $\mathscr{W}$ -assignment and  $g$  a  $\mathscr{W}$ -concept, then

$$\llbracket \phi \rrbracket_{A[g \triangleright x]} \Rightarrow \llbracket \exists x \phi \rrbracket_A.$$

**Proof.**

Suppose  $w \in \llbracket \phi \rrbracket_{A[g \triangleright x]}$ . Then  $\mathcal{M}^w, A[g \triangleright x]^w \models \phi$ , so  $\mathcal{M}^w, A^w[g(w) \triangleright x] \models \phi$ , and hence  $\mathcal{M}^w, A^w \models \exists x \phi$ . Thus  $w \in \llbracket \exists x \phi \rrbracket_A$ . ◻

Notice that, while this is close to proposition 19, it doesn't quite follow from it, because in proposition 19 both  $\mathscr{W}$ -expressions were relativized to the same variable assignment, and here they are not.

### 4.5 Terms, Denoting, and Expressing

Model theory uses the concept of a term's (name or variable's) *denotation* on a model, relative to a variable assignment. This is just whatever that term picks out, relative to that assignment. Likewise, we can have the concept of what a term (name or variable) *expresses* on an intensional space  $\mathscr{W}$ , relative to some  $\mathscr{W}$ -assignment. This will be the  $\mathscr{W}$ -concept that the term expresses. If the term  $\alpha$  is a name, it will be  $\llbracket \alpha \rrbracket_{\mathscr{W}}$  — the  $\mathscr{W}$ -intension of the name. When  $\alpha$  is a variable, this will be the  $\mathscr{W}$ -concept  $\alpha$  is assigned to. More precisely,

**Definition 21.**

Let  $\mathscr{W}$  be an intensional space over  $\mathcal{L}$ ,  $\alpha$  a term of  $\mathcal{L}$ , and  $A$  a  $\mathscr{W}$ -assignment.

$$\llbracket \alpha \rrbracket_{\mathscr{W}, A} = \begin{cases} \llbracket \alpha \rrbracket_{\mathscr{W}} & \text{if } \alpha \text{ is a name and} \\ A(\alpha) & \text{if } \alpha \text{ is a variable.} \end{cases}$$

As with other such definitions we drop the ' $\mathscr{W}$ ' when no confusion arises. Now we can show a substitution principle, which says, in effect, that that the proposition expressed by  $\phi(\vec{\alpha})$  is the same as the one expressed by  $\phi(\vec{x})$  when each variable  $x_i$  is assigned to  $\alpha_i$ 's intension.<sup>6</sup>

**Lemma 22.**

$$\llbracket \phi(\vec{x}) \rrbracket_{A[\llbracket \alpha \rrbracket_A \triangleright \vec{x}]} = \llbracket \phi(\vec{\alpha}) \rrbracket_A.$$

<sup>6</sup>This would admit of counterexamples if some  $x$  in  $\vec{x}$  occurred free in  $\phi(\vec{\alpha})$ ; but since the latter replaces *every* variable in  $\vec{x}$  for some  $\alpha$ , this isn't a problem.

**Proof.**

For this, note that if  $\alpha$  is a variable and  $w \in W$ ,  $(\alpha)_A(w) = A^w(\alpha) = \alpha^{\mathcal{M}^w, A^w}$ , and if  $\alpha$  is a constant,  $(\alpha)_A(w) = I^{\mathcal{M}^w}(\alpha) = \alpha^{\mathcal{M}^w, A^w}$ .

$$\begin{aligned}
w \in \llbracket \phi(x) \rrbracket_{A[\overrightarrow{(\alpha)}_A \triangleright \vec{x}]} &\text{ iff } \mathcal{M}^w, A[\overrightarrow{(\alpha)}_A \triangleright \vec{x}]^w \models \phi(\vec{x}) \\
&\text{ iff } \mathcal{M}^w, A^w[\overrightarrow{(\alpha)}_A(w) \triangleright \vec{x}] \models \phi(\vec{x}) \\
&\text{ iff } \mathcal{M}^w, A^w[\overrightarrow{\alpha^{\mathcal{M}^w, A^w}} \triangleright \vec{x}] \models \phi(\vec{x}) \\
&\text{ iff } \mathcal{M}^w, A^w \models \phi(\vec{\alpha}) \\
&\text{ iff } w \in \llbracket \phi(\vec{\alpha}) \rrbracket_A
\end{aligned}$$

The third biconditional relies on the identity noted above, and the penultimate one on the basic model-theoretic fact that  $\mathcal{M}, a[\alpha^{\mathcal{M}, a} \triangleright x] \models \phi(x)$  iff  $\mathcal{M}, a \models \phi(\alpha)$ .  $\dashv$

Lemma 22 has a number of immediate corollaries. For instance, if two formulas differ only by reshuffling of the variables, and two  $\mathscr{W}$ -assignments differ only in ways that compensate for those variable-shufflings, then the first formula will express the same thing on the first assignment that the second formula expresses on the second.

#### 4.6 What a Propositional Theory ‘Says’

Everything so far has just been about  $\mathscr{W}$ -propositions and what sentences express them. But we can also ask how these notions relate to particular propositional theories. Suppose, for instance, that  $T$  is a set of  $\mathscr{W}$ -propositions and includes  $\llbracket \text{Philosopher(jason)} \rrbracket_{\mathscr{W}}$ . If so, there is a very clear sense in which the theory ‘says’ that Jason is a philosopher.

The scare quotes are important.  $\llbracket \text{Philosopher(jason)} \rrbracket_{\mathscr{W}}$  is just some old set of things we call ‘worlds’, picked out by some fancy set-theoretic apparatus. If there are genuine propositions, and they are sets of genuine possible worlds, there is no reason to think  $\llbracket \text{Philosopher(jason)} \rrbracket_{\mathscr{W}}$  is one of them, and much less the one that counts as the proposition that Jason is a philosopher. But insofar as we pretend that the ‘worlds’ in  $W$  are genuine possible worlds, and pretend that  $\mathcal{M}^w \models \phi$  if and only if  $\phi$  is true in the world  $w$ , then we will want to pretend that  $\llbracket \text{Philosopher(jason)} \rrbracket_{\mathscr{W}}$  is the proposition that Jason is a philosopher. Since the pretense helps us understand the underlying model theory, we’ll stick with it here.

Let  $j$  be the  $\mathscr{W}$ -concept associated with ‘jason’ — that is, let  $j = \llbracket \text{jason} \rrbracket_{\mathscr{W}}$  — but now imagine we remove the name ‘jason’ from our language. In this case, we can’t call  $j$  ‘ $\llbracket \text{jason} \rrbracket_{\mathscr{W}}$ ’ anymore. But the  $\mathscr{W}$ -proposition which ‘says’ that Jason is a philosopher is still hanging around. We can even say which one it is, by using variable assignments. Let  $A$  be any variable assignment; then  $A[j \triangleright x]$  assigns the  $\mathscr{W}$ -concept  $j$  to  $x$ . In this case,

$$\llbracket \text{Philosopher}(x) \rrbracket_{A[j \triangleright x]}$$

is the  $\mathscr{W}$ -proposition which says that Jason is a philosopher. More precisely, it is the very same set of worlds that we identified with  $\llbracket \text{Philosopher}(\text{jason}) \rrbracket_{\mathscr{W}}$  in the expanded language.

Notice that, thanks to proposition 22, we will always have

$$\llbracket \text{Philosopher}(x) \rrbracket_{A[j \triangleright x]} = \llbracket \text{Philosopher}(y) \rrbracket_{A[j \triangleright y]}$$

for any variables  $x$  and  $y$ . So, in a way, the particular choice of variable doesn't matter for talking about this set. In order to streamline notation, rather than write the  $\mathscr{W}$ -proposition this way, we will write it as

$$\llbracket \text{Philosopher}(\underline{j}) \rrbracket_A$$

. We underline the  $j$  to highlight the fact that it isn't a name in our object language, but rather a  $\mathscr{W}$ -concept that is 'filling in a blank' — a variable position — in a formula.

More generally, we define, for any  $\mathscr{W}$ -concept  $g$  and formula  $\phi(x)$ ,

$$\llbracket \phi(\underline{g}) \rrbracket_A = \llbracket \phi(x) \rrbracket_{A[g \triangleright x]}.$$

This trick can be stacked multiple times. For instance, if we want to talk about propositions which 'say' that two  $\mathscr{W}$ -concepts,  $g$  and  $h$ , are identical, we use  $\llbracket \underline{g} = \underline{h} \rrbracket_A$ , and unpack it as follows:

$$\llbracket \underline{g} = \underline{h} \rrbracket_A = \llbracket \underline{g} = y \rrbracket_{A[h \triangleright y]} = \llbracket x = y \rrbracket_{A[h \triangleright y][g \triangleright x]}.$$

This is mainly for convenience; the left-hand side is more intuitive, and easier to read, than the right-hand side.

One nice feature of this notation is that we can rewrite lemma 22 as

$$\llbracket \phi(\vec{\alpha}) \rrbracket_A = \llbracket \phi(\overrightarrow{\llbracket \alpha \rrbracket_A}) \rrbracket.$$

## 5 ŁOS'S THEOREM

All of this work has a point. We are going to prove that, if a set of sentences corresponds, in the right way, to an ultrafilter, then it has a model. More precisely, if  $\mathscr{W}$  is an intensional space over  $\mathcal{L}$  and  $T$  an ultrafilter over  $\mathscr{W}$ , then there is a model that makes  $\phi$  true if and only if  $\llbracket \phi \rrbracket_{\mathscr{W}}$  is in  $T$ .

Before describing the proof, let's remind ourselves of Henkin's (1949) technique for proving completeness, which has now become the standard textbook method. Henkin's proof shows that if some theory is syntactically consistent, it has a model. We start with a syntactically consistent theory, and then expand it to a consistent theory that is negation-complete (via Lindenbaum's lemma) and includes every instance of a witness axiom. We then build a model *out of* this theory: Roughly, the constants become the model's domain (with the witness axiom ensuring there are enough constants to go around) and we read off extensions' predicates from

the theory itself. For instance, if  $'Fc'$  is in the theory, we put  $'c'$  in the extension of  $'F'$ , and if  $'\sim Fd'$  is in the theory, we keep  $'d'$  out.

The proof of Łos's theorem is, in many ways, the propositional counterpart of this Henkin-style completeness proof. We start with a filter  $T^-$  on an intensional space  $\mathscr{W}$ , which is our propositional analogue of a consistent theory. First we extend it to an ultrafilter  $T$  using the Central Theorem on Ultrafilters, which is our propositional analogue of Lindenbaum's Lemma. Then, instead of using constants, we build our model out of  $\mathscr{W}$ -concepts. Roughly, the  $\mathscr{W}$ -concepts make up the model's domain. (We need no witness axiom to ensure there are enough this time; the nature of  $\mathscr{W}$ -concepts does this for us.) And we read predicates' extensions from the theory itself: if  $T$  'says' that  $g$  satisfies  $'F'$ , then we put  $g$  in the extension of  $'F'$ , and if  $T$  'says' that  $h$  does not satisfy  $'F'$ , then we keep  $g$  out.

If the language has an identity predicate, complications arise. In the case of Henkin's proof, the problem is that  $c$  and  $d$  may be different constants, but the theory may include the sentence  $'c = d'$ . Given the way the meaning of  $'='$  is hardwired into the model, if  $c$  and  $d$  are two separate entities in the domain, the model will not be able to make  $'c = d'$  true. In our case, the problem is similar.  $\llbracket g = h \rrbracket$  may be in  $T$ , even if  $g$  and  $h$  are different functions. Given the way the meaning of  $'='$  is hardwired into the model, if we build our model by putting  $g$  and  $h$  in separately, we will have a hard time making sure the right formulas come out true on the right variable assignments.

The solution in both cases is the same: Use equivalence classes. In Henkin's proof, we call two constants  $c$  and  $d$  equivalent if the maximal consistent set includes  $'c = d'$ . In Łos's proof, we say that  $g$  and  $h$  are equivalent if  $\llbracket g = h \rrbracket \in T$ . Once we do this, then rather than making our domains out of constants or  $\mathscr{W}$ -concepts, we make them out of *equivalence classes* of them.

## 5.1 Building the Model

Enough with comparisons. Let's get to model-building. It turns out that the model we build doesn't depend on  $T$ 's being an *ultrafilter*, but just on its being a *filter*. When it's a filter, the model is called a **reduced product**, and when the filter is also ultra, it's called an **ultraproduct**.

Let's start with our equivalence relation.

### Definition 23.

Let  $\mathscr{W}$  be an intensional space and  $T$  a filter on  $\mathscr{W}$ . Then, if  $g$  and  $h$  are  $\mathscr{W}$ -individual concepts,  $g \equiv_{T, \mathscr{W}} h$  iff  $\llbracket g = h \rrbracket_{\mathscr{W}} \in T$ .

In general we know what the relevant  $T$  and  $\mathscr{W}$  are, and suppress their subscripts to reduce clutter.

With the resources of §4, it isn't difficult to verify that  $\equiv$  is reflexive, transitive, and symmetric, and therefore an equivalence relation. For instance, when it comes to symmetry, the fact that  $x = y \vDash y = x$  combines with proposition 19 plus and

implication closure to tell us that  $\llbracket \underline{g} = \underline{h} \rrbracket \Rightarrow \llbracket \underline{h} = \underline{g} \rrbracket$ , so by implication closure, if  $\llbracket \underline{g} = \underline{h} \rrbracket$  is in  $T$ , so is  $\llbracket \underline{h} = \underline{g} \rrbracket$ .

For transitivity, if  $\llbracket \underline{g} = \underline{h} \rrbracket \in T$  and  $\llbracket \underline{h} = \underline{j} \rrbracket \in T$ , we use closure under conjunction to conclude that  $\llbracket \underline{g} = \underline{h} \wedge \underline{h} = \underline{j} \rrbracket \in T$ , and then and implication closure plus proposition 19 to conclude that  $\llbracket \underline{g} = \underline{j} \rrbracket \in T$ .

Once we've done that, we can define equivalence classes over  $\mathscr{W}$ -concepts:  $\llbracket g \rrbracket_{T, \mathscr{W}}$  is the  $\equiv_{T, \mathscr{W}}$ -class that contains  $g$ . Again we'll drop subscripts whenever we can get away with it. Now, finally, we can define our model.

**Definition 24.**

Let  $\mathscr{W}$  be an intensional space and  $T$  a filter on  $\mathscr{W}$ . Then the **reduced product** of  $\mathscr{W}$  on  $T$  is the model  $\mathcal{M}^T = \langle D^T, I^T \rangle$  where

- (i)  $D^T = \{ \llbracket g \rrbracket : g \text{ is a } \mathscr{W}\text{-individual concept} \}$ ,
- (ii)  $I^T(\alpha) = \llbracket \llbracket \alpha \rrbracket_{\mathscr{W}} \rrbracket$ , and
- (iii)  $I^T(\Pi) = \{ \llbracket \vec{g} \rrbracket : \llbracket \Pi \vec{g} \rrbracket \in T \}$ .

Mathematicians write this model as

$$\prod_{w \in W} \mathcal{M}^w / T,$$

but to keep notation compact I'll stick with  $\mathcal{M}^T$ .

If this definition is to do its job, it had better be legit. Parts (i) and (ii) are unproblematic, but (iii) only makes sense if the following holds:

**Proposition 25.**

If  $g_i \equiv h_i$  for each  $i$ , then  $\llbracket \Pi \vec{g} \rrbracket \in T$  iff  $\llbracket \Pi \vec{h} \rrbracket \in T$ .

If proposition 25 didn't hold, then clause (iii) could give us contradictory instructions, for we could have  $\llbracket \Pi \vec{g} \rrbracket \in T$ ,  $\llbracket \Pi \vec{h} \rrbracket \notin T$  and  $\vec{g} = \vec{h}$ . Then clause (iii) would tell us to both put this sequence in  $\Pi$ 's extension and to leave it out, which of course we can't do. Fortunately, though, this can't happen.

**Proof.**

Suppose  $g_i \equiv h_i$  for each  $i$ . Then  $\llbracket \underline{g}_i = \underline{h}_i \rrbracket \in T$  for each  $i$ . Suppose further that  $\llbracket \Pi \vec{g} \rrbracket \in T$ . Since  $T$  is closed under conjunction,

$$\llbracket \Pi \vec{x} \wedge \underline{g}_1 = \underline{h}_1 \wedge \dots \wedge \underline{g}_n = \underline{h}_n \rrbracket \in T.$$

Let  $A$  be a variable assignment where  $A(x_i) = g_i$  and  $A(y_i) = h_i$  for each  $i$ . Then

$$\llbracket \Pi \vec{x} \wedge x_1 = y_1 \wedge \dots \wedge x_n = y_n \rrbracket_A \in T.$$

However, we have that

$$\Pi \vec{x} \wedge x_1 = y_1 \wedge \dots \wedge x_n = y_n \vDash \Pi \vec{y}.$$

So by proposition 19 and implication closure,

$$\llbracket \Pi \vec{y} \rrbracket_A \in T.$$

Given what  $A$  assigns each  $y$  to, this is definitionally equivalent to what we wanted to show. The other direction is precisely the same.  $\dashv$

## 5.2 The Proof

We now have our model  $\mathcal{M}^T$ . To state Łos's theorem, we need one more piece of machinery. Where  $A$  is any  $\mathscr{W}$ -assignment, we let  $[A]$  be the assignment on  $\mathcal{M}^T$  where  $[A](x) = [A(x)]$ . In other words, if  $A$  assigns some  $\mathscr{W}$ -concept to  $x$ , then  $[A]$  assigns its equivalence class to  $x$ . Now we can prove:

### Theorem 3. Łos's Theorem

If  $T$  is an ultrafilter over a  $\mathscr{W}$ -intensional space and  $\mathcal{M}^T$  its ultraproduct, then for any  $\mathscr{W}$ -assignment  $A$ ,

$$\mathcal{M}^T, [A] \vDash \phi \text{ iff } \llbracket \phi \rrbracket_A \in T.$$

As with the Central Theorem, this proof requires the axiom of choice.<sup>7</sup> We start with

### Lemma 26.

If  $A$  is a  $\mathscr{W}$ -assignment, then

- (i)  $\overrightarrow{\alpha^{\mathcal{M}^T, [A]}} \in I^T(\Pi)$  iff  $\llbracket \Pi \vec{\alpha} \rrbracket_A \in T$ , and
- (ii)  $\alpha^{\mathcal{M}^T, [A]} = \beta^{\mathcal{M}^T, [A]}$  iff  $\langle \alpha \rangle_A \equiv \langle \beta \rangle_A$ .

### Proof.

Note that for any term  $\alpha$ ,  $\alpha^{\mathcal{M}^T, [A]} = \langle \alpha \rangle_A$ . Call this (ID).

For (i):  $\overrightarrow{\alpha^{\mathcal{M}^T, [A]}} \in I^T(\Pi)$  iff (by ID)  $\overrightarrow{\langle \alpha \rangle_A} \in I^T(\Pi)$  iff (by construction)  $\llbracket \Pi \overrightarrow{\langle \alpha \rangle_A} \rrbracket \in T$  iff (by lemma 22)  $\llbracket \Pi \vec{\alpha} \rrbracket_A \in T$ .

For (ii):  $\alpha^{\mathcal{M}^T, [A]} = \beta^{\mathcal{M}^T, [A]}$  iff (by ID)  $\langle \alpha \rangle_A = \langle \beta \rangle_A$  iff  $\langle \alpha \rangle_A \equiv \langle \beta \rangle_B$ .  $\dashv$

Now we are in a position to prove Łos's theorem. The proof is by induction on the complexity of formulas.

<sup>7</sup>Łos's theorem is weaker than choice. However, the conjunction of Łos's theorem and the Central Theorem is equivalent to choice. See Howard 1975.

**Proof.**

The base case is immediate by lemma 26.

For the induction step, we suppose that for any  $\psi$  less complex than  $\phi$ ,  $\mathcal{M}^T, [A] \models \psi$  iff  $\llbracket \psi \rrbracket_A \in T$ .

For negation,  $\mathcal{M}^T, [A] \models \sim\psi$  iff  $\mathcal{M}^T, A \not\models \psi$  iff (by the induction hypothesis)  $\llbracket \psi \rrbracket_A \notin T$  iff (since  $T$  is an ultrafilter)  $-\llbracket \psi \rrbracket_A \in T$  iff  $\llbracket \sim\psi \rrbracket_A \in T$ .

For conjunction,  $\mathcal{M}^T, [A] \models \psi \wedge \chi$  iff  $\mathcal{M}^T, [A] \models \psi$  and  $\mathcal{M}^T, [A] \models \chi$  iff (by the induction hypothesis)  $\llbracket \phi \rrbracket_A \in T$  and  $\llbracket \psi \rrbracket_A \in T$  iff (since  $T$  is a filter\* and so meets conditions (i\*) and (ii\*))  $\llbracket \phi \rrbracket_A \cap \llbracket \psi \rrbracket_A \in T$  iff  $\llbracket \phi \wedge \psi \rrbracket_A \in T$ .

For quantification, we go in two directions. First, suppose that  $\mathcal{M}^T, [A] \models \exists x\psi$ . Then for some  $\mathscr{W}$ -concept  $g$ ,  $\mathcal{M}^T[A](\llbracket g \triangleright x \rrbracket) \models \psi$ , so  $\mathcal{M}^T, [A(g \triangleright x)] \models \psi$ . By the induction hypothesis,  $\llbracket \psi \rrbracket_{A[g \triangleright x]} \in T$ . Thus by proposition 20 and  $T$ 's closure under implication,  $\llbracket \exists x\psi \rrbracket_A \in T$ .

For the other direction, suppose  $\llbracket \exists x\psi \rrbracket_A \in T$ . Since  $T$  is a filter this proposition is not empty. Let  $w \in \llbracket \exists x\psi \rrbracket_A$ ; then  $\mathcal{M}^w, A^w \models \exists x\psi$ , so for some  $e_w \in D^w$ ,  $\mathcal{M}^w, A^w[e_w \triangleright x] \models \psi$ . Let  $g(w) = e_w$  for each  $w \in \llbracket \exists x\psi \rrbracket_A$ . (Such a  $g$  is guaranteed by the axiom of choice.) Then each  $w \in \llbracket \psi \rrbracket_{A[g \triangleright x]}$ . By the induction hypothesis,  $\mathcal{M}^T, [A[g \triangleright x]] \models \psi$ , so  $\mathcal{M}^T, [A][\llbracket g \triangleright x \rrbracket] \models \psi$ , so  $\mathcal{M}^T, [A] \models \exists x\psi$ .  $\dashv$

Łos's theorem is powerful. It tells us that if we have an intensional space and a  $\mathscr{W}$ -propositional theory  $T$  in that intensional space, if  $T$  is an ultrafilter, then it has a model — namely, the ultraproduct. And if  $T$  is merely a filter, we can first use the Central Theorem on Ultrafilters to expand it and get a model that makes all of  $T$  true plus more besides.

## 6 APPLICATIONS

So what can a *philosopher* do with these resources? This isn't the place to delve into the details, but here are a few places where ultrafilters crop up in philosophy.

### 6.1 Compactness

Many philosophically-minded logic texts take a moment to prove:

**Theorem 4. Compactness**

If every finite subset of  $\Gamma$  has a model, then so does  $\Gamma$ .

The usual method goes through completeness and proof theory:  $\Gamma$  has no model iff it can be used to derive a contradiction iff some finite subset can be used to derive a contradiction iff that finite subset has no model. But the theorem itself says nothing about derivations, and seems as though it should admit a purely model-theoretic proof.

Łos's theorem provides one. If every finite subset of  $\Gamma$  has a model, then we can create an intensional space  $\mathscr{W}$  with worlds, one for each of these finite subsets.

Next, we consider all the  $\mathscr{W}$ -propositions that  $\mathscr{W}$ -express finite subtheories of  $\Gamma$ , and call this theory  $F^-$ . We can show that  $F^-$  is closed under conjunction, and it clearly doesn't contain the empty set, since each finite subset of  $\Gamma$  has a model. So we can take the filter  $F$  based on  $F^-$ ; we know it's a filter by proposition 7. Then we use the Central Theorem to extend this to an ultrafilter  $U$ , and take its ultraproduct  $\mathcal{M}^U$ .

Now take any  $\phi \in \Gamma$ . It corresponds to a finite subset of  $\Gamma$  (namely,  $\{\phi\}$ ) so  $\llbracket \phi \rrbracket_{\mathscr{W}}$  is in  $U$ . So by Łos's theorem,  $\mathcal{M}^U \models \phi$ . Since  $\phi$  was arbitrary,  $\mathcal{M}^U$  makes true *every* member of  $\Gamma$ , and so is a model of  $\Gamma$ .

The only tricky bit is showing that  $F^-$  is closed under conjunction. But if  $p$  and  $q$  both express some finite subtheory of  $\Gamma$ , then they will be  $\llbracket \Delta \rrbracket_{\mathscr{W}}$  and  $\llbracket \Sigma \rrbracket_{\mathscr{W}}$  for some finite subtheories  $\Delta$  and  $\Sigma$ . In this case,  $\llbracket \Delta \wedge \Sigma \rrbracket_{\mathscr{W}}$  will also express a finite subtheory of  $\Gamma$  (namely,  $\Delta \cap \Sigma$ ) and so be in  $F^-$ . But this  $\mathscr{W}$ -proposition is just  $p \wedge q$ . So we have our closure under conjunction, and Compactness follows.

## 6.2 Minimal Modal Logic

The modal logics we inherited from Saul Kripke (1959) are **normal**: They validate the axiom

$$\mathbf{(K)} \quad \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$$

and the inference rule

$$\mathbf{(Nec)} \quad \text{If } \models \phi, \text{ then } \models \Box\phi.$$

These are plausible for alethic modalities, but may not be appropriate when philosophers try to adapt modal logic for other purposes.

Suppose we wanted to interpret ' $\Box$ ' as 'it ought to be the case that', for instance. Classic modal logics have a modal explosion theorem,

$$(\Box\phi \wedge \Box\sim\phi) \rightarrow \Box\psi.$$

We might think that Sartre's (1946) student both ought to join the resistance and also not join the resistance (so he can stay home and support his ailing mother), but even if we think that we do *not* think it is plausible that he join the Nazi's. Normal modal logics are too strong for such an application.

Neighborhood semantics, or 'Scott-Montague semantics'<sup>8</sup>, have a set  $W$  of possible worlds each of which is assigned a **neighborhood** — a  $W$ -propositional theory. There aren't any constraints on neighborhoods, and we can assign a neighborhood containing both  $p$  and  $\sim p$  without it containing their conjunction, much less any other arbitrary proposition, so we can deal with Sartre's modal dilemmas.<sup>9</sup> But it turns out that a neighborhood model will validate (K) and (Nec) exactly when each of its neighborhoods is a pre-filter, and will further validate the (D) axiom ( $\Box\phi \rightarrow \Diamond\phi$ ) when each of those neighborhoods is a filter. So filters are useful in characterizing, from a neighborhood perspective, what makes a logic normal.

<sup>8</sup>For its developers, Richard Montague (1970) and Dana Scott (1970).

<sup>9</sup>Chellas (1980) contains an extensive discussion of neighborhood semantics, including its relation to deontic logic.

### 6.3 Social Choice Theory

Suppose we are going to hold a vote. We want our voters to rank order some options, and we'll come up with an aggregate order of those options based on the votes. If it's to be sensible, the total order should put  $O_1$  above  $O_2$  whenever every single voter does; and if *they* are to be sensible, the way each voter ranks  $O_1$  and  $O_2$  shouldn't depend on what they think of some third option,  $O_3$ .

Arrow's (1950) famous impossibility theorem shows that, so long as the voters themselves are free to rank the candidates any which way, and there are only finitely many of them, an aggregation procedure can only work this way if one voter is *dictatorial*: Whatever that voter wants, they get.

Here, in sketch, is how the proof works. Think of various properties the voters could have: properties like *being a voter*, *being on the board*, *being a member of the founding family*, and so on. Some of these properties are *decisive*: If everybody with the property votes the same way, the final ranking must agree with the way these voters voted.

We can think of properties as sets of individuals, and higher-order properties — such as *being decisive* — as sets of properties. But, like propositions, properties can be conjoined, have negations, imply others, and so on. So some higher-order properties can be filters or ultrafilters.

Arrow's theorem shows that, if our aggregation procedure has all the sensible features we want, the higher-order property of *being decisive* is an ultrafilter. But we only have finitely many voters. So by proposition 11, some singleton set — in other words, some heccaetistic property, such as *the property of being so-and-so* — will be decisive. In that case, so-and-so is a dictator.

Having set up the problem in terms of filters and ultrafilters, we can use them to see whether it can be mitigated, as Austen-Smith and Banks (1998) do. They show, for instance, that if we allow our aggregate ranking to be quasi-transitive, the property of *being decisive* will be filter, but not necessarily an ultrafilter, in which case we have not a dictator but an oligarchy.

### 6.4 Infinitesimals

When developing the calculus, both Leibniz and Newton, at some point, relied on the idea of an **infinitesimal** value — a value smaller than any positive real number, but still greater than zero. Eventually Newton gave up on the idea, but Leibniz never did. The idea suffered from a barrage of criticism, though. Bishop Berkeley decried infinitesimals as “ghosts of departed quantities” (1734), and in the *Principia* Russell called them “unnecessary, erroneous, and self-contradictory.”<sup>10</sup>

Nonetheless, the appeal of infinitesimals has persisted, both as a mathematical concept in its own right and as a philosophical tool. On the usual real-valued understanding of probability, when I throw a dart with a point-sized tip at a dartboard, it is guaranteed to land somewhere that it has zero chance of landing. Anyone who think “zero chance” means “impossible” cannot accept this. Given

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<sup>10</sup>See Bell 2022: §§4–5 for detailed historical discussion.

infinitesimals, though, the chance of hitting any particular point needn't be zero — it need only be infinitesimal.<sup>11</sup>

Arguments from usefulness may be little better than wishful thinking. The best way to show that some mathematical entities are in good standing is to just *construct* the damn things. Negative, irrational, and complex numbers have all been held suspect at some point in their lives, and in each case the skepticism ultimately fell to construction.

With infinitesimals, nobody knew how to construct them until Abraham Robinson came along in the 1960s. His (1966) method uses Łos's theorem. What we do is first construct an intensional space  $\mathscr{W}$  where the worlds are the natural numbers and each world's associated model is a model of the real numbers. The  $\mathscr{W}$ -concepts of this intensional space then end up being functions from natural numbers to real numbers. We can think of these functions just as well as *infinite sequences* — a  $\mathscr{W}$ -concept  $g$  in this intensional space is the infinite sequence whose  $n$ -th entry is  $g(n)$ .

We then take a non-principal ultrafilter of natural numbers and construct the ultraproduct of this intensional space. (When each world has the same model, this ultraproduct is called an **ultrapower**.) Its domain contains equivalence classes of sequences, and two sequences count as equivalent when they differ on at most finitely many entries.<sup>12</sup> The elements of this model are called **hyperreals**.

Here's an intuitive way to think about how this works. Inspired by Berkeley's talk of numerical 'gosts', imagine abstract objects that coincide with real numbers, but at different possible worlds may coincide with *different* real numbers. Then we take our ultrafilter over these possible worlds. We call two ghosts  $g$  and  $h$  **siblings** if the set of worlds where they coincide is in the ultrafilter. The sibling relation is an equivalence relation, and we call a maximal set of equivalent siblings a **family** of ghosts. Next, say that two ghosts **rarely part ways** when the set of worlds where they don't coincide is finite. Since we used a non-principal ultrafilter to define siblings, two ghosts will be siblings when they rarely part ways.<sup>13</sup> These families of ghosts correspond to the hyperreals.

Some ghosts coincide with the same real number at every world. Their family is a *real* family — it is a family that represents a real number in the hyperreals. But there are infinitesimal families, too — families that count as less than any positive real family, but greater than the zero family.

Consider the function  $g(n) = \frac{1}{n}$  — this is the ghost that, at world  $n$ , coincides with the fraction  $\frac{1}{n}$ . Now choose any positive real ghost  $r$ , and the real ghost  $0$ . If we imagine the possible worlds as real lines, these two ghosts cut straight lines across modal space. But the ghost  $g$  doesn't cut a straight line — he cuts a line that starts at one and then gets closer and closer to zero the further into modal space we go. At some point,  $g$  will start coinciding with numbers smaller than  $r$ ,

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<sup>11</sup>Cf. Skyrms 1980 for a developed defense of a view in this vein. For criticism, see Easwaran 2014.

<sup>12</sup>Because it's an ultrafilter, this isn't the only way two sequences can count as equivalent, but it's the most useful one.

<sup>13</sup>There will also be some siblings that *do* rarely part ways.

and will continue coinciding with smaller numbers throughout the rest of modal space. This means that even if for the first (finitely many) worlds,  $r(w)$  is smaller than  $g(w)$ , for the rest  $g(w)$  is smaller than  $r(w)$ . (See figure 2.) Since there are

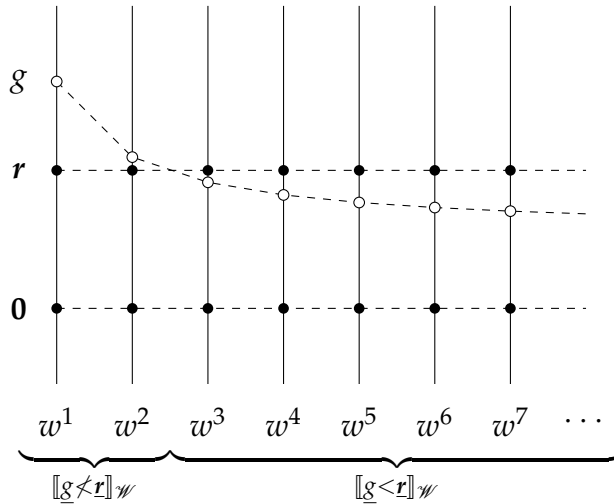


Figure 2: The Shrinking Ghost  $g$

only finitely many worlds where  $g$  is not smaller than  $r$ ,  $[[\mathbf{0} < g < r]]_{\mathscr{W}}$  is rarely false, and so in our ultrafilter. This means, in the ultrapower, the  $g$ -family counts as greater than the  $\mathbf{0}$ -family and larger than any positive real  $r$  family. Thus the  $g$ -family is infinitesimal.

## 7 CONCLUSION

Filters and ultrafilters can be powerful tools in the hands of technically-minded philosophers. We often find ourselves in situations where we have some gizmos — propositional theories, higher-order properties, or what-have-you — that are closed under conjunction and implication. And we often find ourselves needing to construct a model to show that such-and-such theory is coherent. Despite their unfamiliarity, filters, ultrafilters, and Łos's theorem can help us in these cases.<sup>14</sup>

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